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Nonequilibrium Thermodynamics of Lasing and Bistable Optical Systems[1]

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A simple model of nonlinear optical systems exhibiting instability—such as in laser action and in bistable absorption—is presented that provides a prototype of nonequilibrium thermodynamics on a statistical basis. The adiabatic reduction of the atomic degrees of freedom in a revised Langevin treatment establishes a fully consistent framework in which the active electromagnetic field mode is in contact with two thermal reservoirs (the cavity and the atoms) as well as being acted upon by an external field. The results are summarized by the first and second laws, $dE = \delta W + \delta Q_c + \delta Q_A$ and $dS \geqslant \delta Q_c/\bar{T}_c + \delta Q_A/\bar{T}_A$, with the statistical mechanical representations of the entities therein exhibiting the nature of the mode; i.e., (a) a heat-engine structure operating between two reservoirs of temperatures $\bar{T}_c > 0$ and $\bar{T}_A < 0$ for the laser, and (b) a nonlinear response against external work balanced by a single reservoir ($\bar{T}_c = \bar{T}_A$) for the absorptive bistability.

KEY WORDS: Lebowitz model of many-reservoir open systems; stochastic calculus; detailed balance; heat engine; nonlinear Onsager coefficient.

1. INTRODUCTION

In a previous paper[1] (hereafter referred to as I), we discussed how a statistical theory of dissipative dynamics can be converted into its thermodynamic description, considering in particular a laser system. This system is presumably the simplest possible example of the instability of dynamics with only a few degrees of freedom, and has been the subject of many studies (it is in fact the first example and prototype of Haken's synergetics[2,3]). To our

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knowledge,[3] however, a full thermodynamic formulation is not available for the macroscopic, steadily streaming aspect of the phenomenon on the basis of microscopic laser theory. The present paper extends I to complete such a formulation: We obtain and discuss thermodynamic relations in situations far from thermal equilibrium involving phase transitions; we include the first-order phase transition induced by an external, coherent field—the so-called optical bistability of recent investigations[(10,11)]—as well as the second-order phase transition induced by the usual pumping, i.e., lasing.

Our approach is not especially novel; the method of Langevin equations was used in the very beginning of laser theory, but is now revised substantially and incorporated into the study of stochastic differential equations (see I). Here, one starts with a set of macroscopic rate equations relevant to the phenomenon and supplements them by stochastic force terms of Gaussian white noise character. It is then possible, in principle, to find the distribution of the system, in particular for the steady state, which provides all the necessary information about the thermodynamics. Investigations of the steady-state distributions for stochastic systems including the solutions of Fokker–Planck equations in the above Langevin approach are quite popular in various branches of statistical physics.

After submitting paper I, we became acquainted with the work of Landauer. He started with a study of fluctuations in bistable tunnel diode circuits,[(12)] and has since published a number of papers[(13–16)] treating various nonlinear systems similar to the laser in an attempt to establish the connection between the steady-state distribution of such a system and its thermodynamic characteristics; in his terminology,[(16)] "$dQ = T\,dS$ far from equilibrium." The present discussion is in agreement with the motivation and starting point of his approach, but we present a more useful form of the thermodynamic relations than his "$dQ = T\,dS$"; we develop our results step by step in subsequent sections. To focus on the point of difference, we cite a paper by Landauer and Woo[(13)] presented in 1972, and especially the discussion of it, where two important questions were asked, one by Fisher, who questioned the formula $\int \rho \log \rho \, dq$ for the entropy, on the basis of which the authors discussed the quantity $T\,dS$, and the other by Van Kampen, who expressed a doubt, which would be rather serious for the entire Langevin approach in the nonlinear regime, concerning the use of the so-called "nonlinear Fokker–Planck equations."

Van Kampen's objection has been given in a concrete statement in his systematic expansion scheme of the master equations,[(17,18)] and the present

[3] The thermodynamics of the laser must be the prototype of thermodynamics involving *negative absolute temperatures*, where a revision of the second law is inevitable[(4–7)] (see the formulation by Nakagomi[(8)]). Such work, however, does not refer to standard laser theory.[(9)] Our main concern is to fill this gap.

approach must be subject to his criticism. A discussion of our standpoint of being based on the nonlinear Langevin equations will be postponed to the last section. Here, we note that there is nothing wrong or inconsistent in the present framework, at least for obtaining the steady-state distributions, which we attribute to the effectiveness of the *stochastic calculus* in conjunction with a consideration of *detailed balance*: it clarifies some confusing points with regard to nonconstant diffusion coefficients.

As to the problem of the relevant introduction of entropy, we point out that there exists a satisfactory theoretical framework due to Lebowitz,[19–22] which we call the *Lebowitz model of many-reservoir open systems*: by means of this model, the concept of entropy production (rather than entropy itself) can be introduced such that the thermodynamic and information-theoretic contributions to it are compatible with each other and also with the second law of thermodynamics. We expect (and in fact verify, though not claiming generality) that a class of nonlinear systems of interest, including the laser and those considered by Landauer, can be adapted to the Lebowitz model. Recently, Spohn and Lebowitz[23] gave a revised version of the model, proposing a formulation of the principle of minimal entropy production. We supplement this formulation by incorporating Prigogine's idea of the "local potential," including the presence of external fields, so that it can be used to characterize the steady state beyond the linear regime of irreversible processes.[24] On this basis, we discuss some thermodynamic aspects of our laser system.

The general thermodynamic relation we deduce is nothing more than the familiar form $dS \geqslant \sum_i (1/T_i)\, \delta Q_i$, but we show that the Lebowitz formulation provides an accurate expression for each entity in this formula and that all the characteristics of the situation far from equilibrium are fully contained in expressions averaged over the distribution. For the steady lasing state, the formula exhibits an ideal *heat engine* operating between two thermal reservoirs, one characterized by a negative and the other by a positive temperature. For the state of absorptive bistability, it is related to a typical *nonlinear Onsager coefficient* of the irreversibility relation, by which the conventional expression of the "entropy production" can be based upon the ensemble-averaged version, where the real meaning of entropy production can be seen.

2. DISSIPATIVE DYNAMICS OF A SYSTEM DRIVEN BY MECHANICAL AND THERMAL CONSTRAINTS

The Brownian motion of a particle in one dimension affected by a potential field and a friction due to the environment—the well-known Ornstein–Uhlenbeck process[25] in a general potential field—will be discussed in this section for the purpose of illustrating the Lebowitz model. If x and u denote the position and the velocity, respectively, of such a Brownian particle of mass m, then the set of Langevin equations governing the motion is given by

$$\frac{dx}{dt} = u \tag{1}$$

$$\frac{du}{dt} = -\gamma u - \frac{1}{m}\frac{\partial \phi}{\partial x} + f_u(t) \tag{2}$$

In the latter equation $\phi(x)$ represents the potential function, γ the friction constant, and $f_u(t)$ the residual fluctuating force, which is assumed to satisfy

$$\langle f_u(t)\rangle = 0, \qquad \langle f_u(0) f_u(t)\rangle = 2D_u\, \delta(t) \tag{3}$$

(i.e., the assumption of stationary Gaussian white noise), as usual. Then, it is known that the constant D_u of the strength of the fluctuation is related to the equilibrium value of the kinetic energy of the particle:

$$D_u = \gamma\langle u^2\rangle_{\mathrm{eq}} = \gamma kT/m \tag{4}$$

namely, the fluctuation dissipation theorem, or, more specifically, the Einstein relation.[25–27]

The probability distribution $p(t; u, x)$ and its steady-state form $p_{\mathrm{st}}(u, x)$ [which is attained by taking $t \to \infty$ of $p(t; u, x)$], over which the time-correlation function of the form $\langle XX(t)\rangle$ is calculated, satisfies the Fokker–Planck equation (specifically, the Kramers equation[27])

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}(-up) + \frac{\partial}{\partial u}\left(\frac{1}{m}\frac{\partial \phi}{\partial x} p\right) + \gamma \frac{\partial}{\partial u}(up) + \frac{\gamma kT}{m}\frac{\partial^2 p}{\partial u^2} \tag{5}$$

In particular,

$$0 = \frac{\partial}{\partial x}(-up_{\mathrm{st}}) + \frac{\partial}{\partial u}\left(\frac{1}{m}\frac{\partial \phi}{\partial x} p_{\mathrm{st}}\right) + \gamma \frac{\partial}{\partial u}(up_{\mathrm{st}}) + \frac{\gamma kT}{m}\frac{\partial^2 p_{\mathrm{st}}}{\partial u^2} \tag{6}$$

Let the potential function $\phi(x)$ be lower bounded and satisfy

$$\phi(x) \to +\infty \qquad \text{for } x \to \pm\infty \tag{7}$$

so that the particle is fully bounded in it. Then, the canonical distribution of the particle in thermal equilibrium

$$p_{\mathrm{eq}}(u, x) \propto \exp\left\{-\frac{1}{kT}\left[\frac{mu^2}{2} + \phi(x)\right]\right\}$$

formally satisfies Eq. (6) because

$$\left(-u\frac{\partial}{\partial x}+\frac{1}{m}\frac{\partial\phi}{\partial x}\frac{\partial}{\partial u}\right)p_{\text{eq}}=0, \qquad \gamma\left(u+\frac{kT}{m}\frac{\partial}{\partial u}\right)p_{\text{eq}}=0 \tag{8}$$

which in general yields a unique steady-state solution of Eq. (6): The property (7) guarantees that the function $e^{-(1/kT)\phi(x)}$ is normalizable in the entire x axis, so that over the equilibrium distribution

$$\begin{aligned} p_{\text{eq}}(u,x) &= \frac{1}{Z(\phi)}\left(\frac{2\pi kT}{m}\right)^{1/2}\exp\left\{-\frac{1}{kT}\left[\frac{mu^2}{2}+\phi(x)\right]\right\} \\ Z(\phi) &= \int_{-\infty}^{\infty}\exp[-\phi(x)/kT]\,dx \end{aligned} \tag{9}$$

the average values $\langle u\rangle_{\text{eq}}$ and $\langle\partial\phi/\partial x\rangle_{\text{eq}}$ vanish, and together with $\langle f_u(t)\rangle = 0$ for all t, the steadiness conditions

$$\frac{d}{dt}\langle x\rangle_{\text{eq}}=0, \qquad \frac{d}{dt}\langle u\rangle_{\text{eq}}=0 \tag{10}$$

are satisfied.

The physical picture behind the above argument is clearly that the Brownian particle, being in contact with a thermal reservoir of a definite temperature T, i.e., affected constantly by the fluctuation force from the reservoir, tends to its equilibrium position determined by the average over the distribution p_{eq}. The stationary, Gaussian white noise characteristic (3) together with the Einstein relation (4) bridge the kinetics and the thermodynamics, ensuring the overall consistency of the argument. We consider two possible modifications of the standard argument of this kind; first, to remove the assumption that the particle is completely bounded in the potential ϕ as indicated in (7), and second, to generalize the thermal reservoir with which the particle is in contact such that it is not restricted to a single reservoir of a definite temperature, but involves more than two of them, each satisfying (3) and (4), but having different γ's and T's.

2.1. First Modification

Let us consider the simplest case, viz. $\partial\phi/\partial x = \text{const}$, $-\infty < x < \infty$. This corresponds to free Brownian motion under the action of a uniform external field $F = -\partial\phi/\partial x$, for which the formal solution $\exp[-(1/kT)(mu^2/2 - Fx)]$ is obviously irrelevant to the equilibrium. There exists a relevant distribution that satisfies the Kramers equation (6) in this case:

$$p_{\rm st}(u, x) = \left(\frac{2\pi kT}{m}\right)^{1/2} \exp\left[-\frac{m}{2kT}\left(u - \frac{F}{m\gamma}\right)^2\right]$$
$$u\frac{\partial p_{\rm st}}{\partial x} = 0, \qquad \frac{\partial}{\partial u}\left[\left(-\frac{F}{m} + \gamma u\right) + \frac{\gamma kT}{m}\frac{\partial}{\partial u}\right]p_{\rm st} = 0 \tag{11}$$

for which

$$\frac{d}{dt}\langle x\rangle_{\rm st} = \langle u\rangle_{\rm st} = \frac{F}{m\gamma} \neq 0 \tag{12}$$

The steady-state distribution $p_{\rm st}$, (11), differs from the equilibrium one $p_{\rm eq}$, (9), with $\phi = 0$, due to the presence of the external field F, which can be considered as a mechanical constraint driving the system from thermal equilibrium. Let us suppose that our system consists of N such particles in a unit volume, each carrying a charge e, and regard the field F as an electric field: $F = eE$. Then Eq. (12) just represents Ohm's law,

$$J = Ne\langle u\rangle_{\rm st} = \sigma E, \qquad \sigma = Ne^2/m\gamma \tag{13}$$

and elementary irreversible thermodynamics[28] tells us that it is associated with a nonvanishing entropy production given by

$$\frac{1}{T}J\cdot E = \frac{1}{T}\sigma E^2 > 0 \tag{14}$$

It will be shown that an information-theoretic construction of entropy production in the spirit of the Lebowitz model agrees with this result. Also, there exists a class of more general potentials such that $\partial\phi/\partial x \neq$ const and yet the particle is not fully bounded. Landauer's example of a tunnel diode circuit belongs to this category, and the steady-state distribution in such a case requires more elaborate treatment (to be given in a separate publication).

2.2. Second Modification

In Eq. (2), we assume that

$$\gamma = \gamma_1 + \gamma_2, \qquad f_u(t) = f_u^{(1)}(t) + f_u^{(2)}(t) \tag{15}$$

and in place of (3) and (4) we have

$$\langle f_u^{(i)}(t)\rangle = 0, \qquad \langle f_u^{(i)}(0) f_u^{(j)}(t)\rangle = 2D_u^{(i)}\,\delta_{ij}\,\delta(t) \tag{16}$$

$$D_u^{(i)} = \gamma_i kT_i/m, \qquad i, j = 1, 2 \tag{17}$$

These imply that the Brownian particle is affected by the fluctuation forces of two independent thermal reservoirs of temperature T_1 and T_2. The associated Fokker–Planck equation (5) is now replaced by

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}(-up) + \frac{\partial}{\partial u}\left(\frac{1}{m}\frac{\partial \phi}{\partial x}p\right) + \frac{\partial}{\partial u}[(\gamma_1 + \gamma_2)up] + \frac{k\gamma_1 T_1 + k\gamma_2 T_2}{m}\frac{\partial^2 p}{\partial u^2} \tag{18}$$

Suppose that the potential ϕ satisfies the binding condition (7). Then it is easy to check that the steady-state solution of (18) is of the form (9) [for which the separate satisfying of p_{eq} in the form (8) holds] but now having a temperature T given by

$$T = (\gamma_1 T_1 + \gamma_2 T_2)/(\gamma_1 + \gamma_2) \tag{19}$$

Thus, the Brownian particle under consideration obeys a steady-state distribution which is still of thermal equilibrium type but is identical to neither of the two equilibria corresponding to the temperatures T_1 and T_2 (unless $T_1 = T_2$). This provides the most elementary example of a Lebowitz-type two-reservoir open system, considered as driven by an external constraint which is not mechanical but thermal: the thermal driving here is evidenced by a heat flow produced from the higher temperature reservoir to the lower temperature one, given by

$$J_h = \frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2} Nk(T_2 - T_1) \tag{20}$$

for the N-particle system. The Lebowitz formulation (see Section 6) then predicts that this is associated with entropy production as follows:

$$J_h\left(\frac{1}{kT_1} - \frac{1}{kT_2}\right) = \frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2}\frac{N}{T_1 T_2}(T_2 - T_1)^2 > 0 \tag{21}$$

The above two examples demonstrate how a nonequilibrium steady state is formed by an external constraint and how its nonequilibrium characteristics can be indicated by entropy production. Although they are in the context of linear irreversible processes, the considerations can be extended to more complex nonlinear systems involving instability and a phase change: This is the state of affairs in a laser system.

3. MACROSCOPIC DESCRIPTION OF A SIMPLE NONLINEAR OPTICAL SYSTEM

The Langevin treatment of the laser rate equation is a substantial chapter in standard laser theory, and here we cite a specific form of the equation of motion for an active mode of the em field from Sargent *et al.*[9] (Chapter 20):

$$\frac{d}{dt}A(t) = -\left[\frac{\nu}{2Q} + i(\Omega - \nu)\right]A(t) + \left[\frac{g^2\mathscr{D}(\omega - \nu)\mathscr{N}}{1 + \mathscr{R}/R_s}\right]A(t) + G(t) \quad (22)$$

If the quantity $G(t)$ is omitted in this equation, it becomes a deterministic rate equation for the amplitude $A(t)$ of the mode, and the inclusion of the noise $G(t)$ makes the process $A(t)$ stochastic. It is derived by eliminating variables of the atomic degrees of freedom (the complex atomic dipole collective mode and the population difference of two atomic levels besides A and A^*) from the starting set of Langevin equations for six random variables. To be self-contained, we outline the derivation within our context in the Appendix, leaving a further account of Eq. (22) to Table I. We obtain the equation for $a(t)$ [$=A(t)$] in the form

$$\frac{da}{dt} = -[\kappa + i(\omega - \nu)](a - \alpha) + \frac{\frac{1}{2}\gamma_{\parallel} s\, d(\nu)\, Z_0}{1 + s(\nu)a^*a}\, a + (\text{noise}) \quad (23)$$

in terms of three frequencies (ω, ω_o, ν), three damping constants (κ, $\gamma_{\perp}$, $\gamma_{\parallel}$), and the saturation factor s [and $s(\nu)$] related to the field–atom coupling parameter g through (N = total number of atoms)

$$s = 4g^2/\gamma_{\parallel}\gamma_{\perp} = O(N^{-1}) \quad (24)$$

$$s(\nu) = s\, d^*(\nu)\, d(\nu), \qquad d(\nu) = \gamma_{\perp}/[\gamma_{\perp} + i(\nu - \omega_0)] \quad (24')$$

Table I. Auxiliary Parameters

Notation[a] This work	Ref. 9	Meaning
ω	Ω	Frequency of the active mode
ω_0	ω	Frequency of the atomic dipole $= (1/\hbar)(E_a - E_b)$
ν	ν	Frequency of the rotating frame $= (\kappa\omega_0 + \gamma_{\perp}\omega)/(\kappa + \gamma_{\perp}) \simeq \omega$ for lasing ($\kappa \ll \gamma_{\perp}$) = frequency of the external field
κ	$\frac{1}{2}\nu Q^{-1}$	Cavity loss
$\gamma_{\perp}$	γ	Relaxation rate for the dipole
$\gamma_{\parallel}$	$(\gamma_a + \gamma_b)/2$	Relaxation rate for the population difference
Z_0	$\frac{1}{2}\mathscr{N}$	Pump parameter [$=\frac{1}{2}(N_a - N_b)$ without saturation] <0 for ordinary, >0 for inverted population
α		Amplitude of coherent external field

[a] γ_a and γ_b denote the relaxation rate of the a (upper) and b (lower) level populations, respectively, of the two active levels in Ref. 9. Because of the difference in the models, however, a different identification $\gamma_{\parallel} = 2\gamma_a\gamma_b/(\gamma_a + \gamma_b)$ in the saturation denominator in Eq. (23) agrees with that, $\mathscr{R}/R_s$, in Eq. (22).

In addition, Eq. (23) contains two externally controllable variables, α (amplitude of the external field) and Z_0 (the pump parameter), defined by

$$Z_0 = \tfrac{1}{2}(N_a - N_b) \tag{25}$$

(N_a = population of the upper level). A more detailed account of the time constants together with the parameters relating to the constraint is given in Table I.

In the presence of an external coherent field α, the amplitude a denotes the total field, which is the sum of the internally generated mode b and α, i.e., $a = b + \alpha$. The external field interacts with the atomic dipoles through the total field but is not subject to the dissipative effect through the cavity, which is expressed in the cavity loss term with damping constant κ $(=\frac{1}{2}\nu Q^{-1})$. Apart from this, the deterministic part of Eq. (23) is essentially equivalent to that of (22). The treatment of the noise part given in standard laser theory, however, is not very satisfactory for the thermodynamic purposes to which our investigation was devoted in I and which we will consider further in the next section. Here, we consider the "balance" of gain and loss terms based on the macroscopic rate equation (23) without the noise part.

Let us combine Eq. (23) with its complex conjugate and write the rate equation for the intensity of the field $a^*a = n$ (photon number in quantum terminology) as follows:

$$\frac{dn}{dt} = 2\,\mathrm{Re}[\kappa + i(\omega - \nu)]\,a\alpha - Cn + \frac{An}{1 + s(\nu)n} \tag{26}$$

where $C = 2\kappa$ $(=\nu/Q$, where Q is the cavity quality factor[9]) and

$$A = \gamma_{\parallel} s(\nu) Z_0 \tag{27}$$

We can interpret the right-hand side of Eq. (26) by the assignments

$2\,\mathrm{Re}[\kappa + i(\omega - \nu)]\,a\alpha$:
rate of work exerted by external field

$-Cn$:
power dissipation of field energy stored in cavity

$An/[1 + s(\nu)n]$:
power emission or absorption of energy by atoms, depending on $A > 0$ or $A < 0$

The steady state is realized, therefore, by the balance among the above three terms, and we classify here two elementary steady states as follows.

3.1. Case of No External Field But Constrained by a Thermal Pump

For this case

$$\alpha = 0, \qquad Cn = \left[\frac{An}{1 + s(\nu)n}\right]_{\nu=\omega} > 0 \tag{28}$$

($Z_0 > 0$: inverted population). A nontrivial root n_s of this equation is given by

$$n_s = \frac{A - C}{s(\omega)C} = \frac{\gamma_\parallel}{2\kappa}(Z_0 - Z_{\text{th}}) = O(N) \tag{29}$$

under the threshold condition

$$Z_0 > Z_{\text{th}} = \frac{\kappa\gamma_\parallel}{2g^2}\left[1 + \frac{(\omega - \omega_0)^2}{\gamma_\perp{}^2}\right] \tag{30}$$

This represents the lasing state (Fig. 1). We indicated in I that the balance between power emission by atoms and power dissipation to the cavity of the field energy expressed in (28) implies the existence of a heat flow in the direction atoms → cavity, given by

$$J_h = An_s/[1 + s(\omega)n_s] > 0 \tag{31}$$

which is extensive [i.e., proportional to the number of atoms N by virtue of the relation (29); note that $s = O(N^{-1})$, $A = O(1)$, $C = O(1)$]. Below

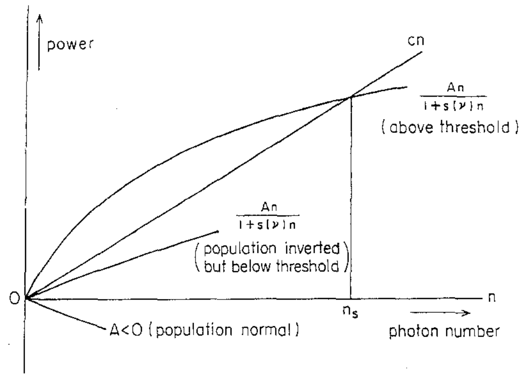

Fig. 1. Illustration of laser instability. The point of intersection n_s of the two power curves on the positive n axis represents the stable fixed point of the laser rate equation above threshold. It is compared with a similar construction of ferromagnetism in mean-field theory, where spontaneous magnetization is predicted to occur as the intersection of the two curves on the magnetization axis below the Curie temperature. (Note, however, the difference between the equilibrium and nonequilibrium natures of the two instability problems: in the laser, the straight line and the intersecting curve have the meaning of "power," i.e., a quantity of dissipation dynamics.)

threshold such a flow is still possible but is only $O(1)$. Therefore, the formation of the lasing state upon the passage of the pump parameter Z_0 through the threshold Z_{th} from below can be characterized by a heat flow that constitutes the order parameter of this second-order phase transition.

3.2. Case of No Thermal Pump But Constrained by an External Field

For this case

$$\mathrm{Re}[\kappa + i(\omega - \nu)]\, a\alpha = Cn + \frac{|A|n}{1 + s(\nu)n}, \qquad A < 0 \tag{32}$$

(Here the frequency ν is interpreted as that of the external field.) The power dissipation of the field energy is both to the cavity and to the atoms, which is totally compensated by the work done by the external field. In this case, a heat flow exists in the direction: external field $\rightarrow$ cavity + atoms. Also, there exists an electric current induced by the total field. For simplicity, we consider the case of complete resonance, $\omega = \omega_0 = \nu$, and use the scaled variables

$$x = s^{1/2}n^{1/2}, \qquad y = s^{1/2}\alpha, \qquad 0 \leqslant x, y < \infty \tag{33}$$

[Since $s = O(N^{-1})$, this scaling makes the fields represented by x and y intensive variables.] The relation between the current $j = \kappa y$ and the total field x is given by either

$$j = \kappa\left(1 + \frac{2c}{1 + x^2}\right)x, \qquad c = \frac{g^2|Z_0|}{\kappa\gamma_\perp} \tag{34}$$

or

$$x = \frac{1}{\kappa}\left(1 + \frac{2c}{1 + x^2}\right)^{-1} j \tag{34'}$$

This is just the equation first discussed by Bonifacio and Lugiato[10] to demonstrate absorptive bistability (Fig. 2), and can be looked upon as a nonlinear irreversibility relation. The nonlinearity causes, for $c > 4$, an interval of y ($y_m \leqslant y \leqslant y_M$) where three allowed values of x exist corresponding to a stable, a metastable, and an unstable point, and the transition from the branch of lower values of x to that of higher values of x can be understood as arising by a saturation of the absorption of the external field when it increases. The treatment of the deterministic equation alone predicts only the existence of such branches but is not able to predict the precise point of the first-order transition where the discontinuity of x takes place.

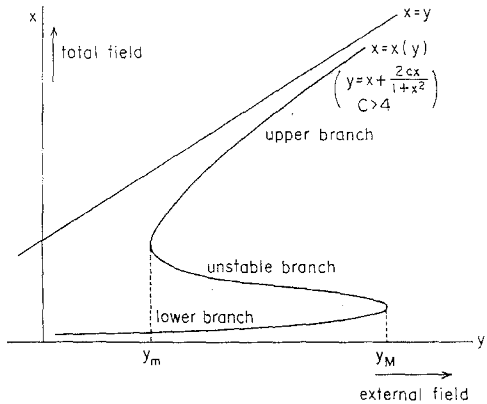

Fig. 2. Illustration of bistable absorption (after Bonifacio and Lugiato[10]). The situation is compared with the hysteresis curve of ferromagnetism, where the abscissa and the ordinate correspond to the external static magnetic field and the magnetization, respectively. Here again, the dissipation dynamic nature of the bistability should be noted.

4. EFFECTS OF NOISE. STEADY-STATE DISTRIBUTIONS

The determination of the probability distribution which describes the stochastic process $a(t)$ subject to the Langevin equation (23) requires the precise form and Gaussian characteristic of the noise part. We specify it in terms of three elementary white noises f, F, and F_z which arise in the starting set of equations (set up in the Appendix):

$$\frac{da}{dt} = -[\kappa + i(\omega - \nu)](a - \alpha) + \frac{\frac{1}{2}\gamma_{\parallel} s\, d(\nu)\, Z_0}{1 + s(\nu)a^*a}\, a + \left(f + \frac{-ig}{\gamma_\perp}\frac{d(\nu)}{1 + s(\nu)a^*a} F + \frac{\frac{1}{2}s(\nu)}{1 + s(\nu)a^*a} aF_z\right) \tag{35}$$

where

$$\langle f^*(0)f(t)\rangle = 2\kappa n_c\, \delta(t), \qquad n_c = \langle b^*b\rangle_{eq} = O(1)$$
$$\langle F^*(0)F(t)\rangle = 2\gamma_{\parallel}\bar{M}\,\delta(t), \qquad \bar{M} = \langle R^*R\rangle_{eq} = O(N) \tag{36}$$
$$\langle F_z(0)F_z(t)\rangle = \gamma_\perp \bar{M}\,\delta(t)$$

and vanishing correlations for all the other combinations.

Equation (35) has been derived by applying the *adiabatic reduction of rapidly relaxing variables* associated with the atomic degrees of freedom, with a formal inclusion of the noise part in the starting Langevin equations. Its validity is assured at least without the noise part, as discussed in the Appendix, provided the condition κ, $|\nu - \omega| \ll \gamma_\perp, \gamma_\parallel$ is satisfied. As to the noise part, on the other hand, the adiabatically reduced form of the white noise in (35) contains a nonconstant factor (and nonlinearity) in the coefficients, which causes a confusion in deducing the correct form of the Fokker–Planck equation for the reduced process $a(t)$. This is due to a singular property of the white noise, which contains "spurious drift."[29] A possible prescription to obtain the correct form will be found by taking as an example the Ornstein–Uhlenbeck process considered in Section 2.

The Ornstein–Uhlenbeck (OU) theory is a fully dynamical treatment of the Einstein theory of Brownian motion such that the solution of (1) and (2) reduces to a spatial diffusion process $x(t)$ asymptotically for $t \to \infty$ (more specifically for $t \gg 1/\gamma$),[26] which can be realized by the adiabatic reduction method, i.e., just by setting $du/dt = 0$ in (2) and by inserting the form of u thus obtained into the right-hand side of (1). If the decay factor γ is a constant, this procedure yields

$$\frac{dx}{dt} = -\frac{1}{m\gamma}\frac{\partial\phi}{\partial x} + f_x(t), \qquad f_x(t) = \frac{1}{\gamma} f_u(t) \tag{37}$$

and the use of (3) and (4) establishes the spatial diffusion process, for which the Fokker–Planck equation is written as

$$\frac{\partial p}{\partial t} = \frac{kT}{m\gamma}\frac{\partial}{\partial x}\left(\frac{1}{kT}\frac{\partial\phi}{\partial x} p + \frac{\partial p}{\partial x}\right) \tag{38}$$

[The potential $\phi(x)$ is assumed to satisfy the binding condition (7).] Suppose, however, that γ (>0) is spatially nonuniform. The procedure then must be reexamined to decide the correct role of $\gamma(x)$ with regard to the second-order differential operation in the Fokker–Planck equation.

To find the answer, we first rewrite the Langevin equations (1) and (2) in the form of the stochastic differential equation

$$dx = u\, dt \tag{39}$$

$$du = \left[-\gamma(x)u - \frac{1}{m}\frac{\partial\phi}{\partial x}\right] dt + \gamma(x)^{1/2}\, dw \tag{40}$$

where the Brownian motion (Wiener process) $w(t)$ satisfies

$$\langle dw(t)\, dw(t)\rangle = (2kT/m)\, dt \tag{41}$$

which conforms to (3) and (4). The prescription to carry out the adiabatic reduction of the velocity process $u(t)$ is now summarized as follows:

(I) The stochastic differential (SD) $\gamma^{1/2}\, dw$ in (40) should be considered in the Itô sense in order for the averaged motion to be consistent with

$$\frac{d}{dt}\langle u\rangle = -\langle \gamma \cdot u\rangle - \frac{1}{m}\left\langle\frac{\partial\phi}{\partial x}\right\rangle$$

(II) Carry out the elimination of the process $u(t)$ by setting $du = 0$ and by inserting $u\,dt$ thus obtained into the right-hand side of (39), where the relation between the Itô and Stratonovich senses of the SD, viz.

$$Y\,dX = Y \circ dX - \tfrac{1}{2}\,dY\,dX$$

($Y \circ dX$ is the Stratonovich sense), is utilized and products of more than three SDs can be discarded.

(III) All the calculus in the above, algebraic and differential, can be performed just as the usual calculus in terms of the Stratonovich sense of the SD; in particular, for any differentiable functions $f(x), g(x),\ldots$ of the process $x(t)$

$$df(x) = \frac{\partial f}{\partial x} \circ dx, \qquad \frac{\partial g}{\partial x} \circ \left(\frac{\partial f}{\partial x} \circ dx\right) = \left(\frac{\partial g}{\partial x}\frac{\partial f}{\partial x}\right) \circ dx$$

A differential form of time $f(t)\,dt$ is considered in the Stratonovich sense.

(IV) Calculate the resulting drift velocity and the diffusion coefficient according to either the Itô or the Stratonovich formula by using (41) to get the Fokker–Planck equation[40]

$$dx = b(x)\,dt + g(x)\,dB \rightarrow \frac{\partial p}{\partial t} = \frac{\partial}{\partial x}(-bp) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(g^2 p) \qquad \text{(I)}$$

$$= \bar{b}(x)\,dt + g(x) \circ dB \rightarrow \frac{\partial p}{\partial t} = \frac{\partial}{\partial x}(-\bar{b}p) + \frac{1}{2}\frac{\partial}{\partial x}\left(g\frac{\partial}{\partial x}gp\right) \qquad \text{(S)}$$

if $B(t)$ is normalized such that $\langle dB\,dB\rangle = dt$.

If these steps are followed in the OU process with nonuniform γ, one finds the Fokker–Planck equation for the reduced $x(t)$ process:

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left[\frac{kT}{m\gamma(x)}\left(\frac{1}{kT}\frac{\partial \phi}{\partial x}p + \frac{\partial p}{\partial x}\right)\right] \qquad (42)$$

In particular, we have the steady-state solution

$$p_{\text{st}}(x) = \frac{1}{Z(\phi)}e^{-\phi(x)/kT}$$

irrespective of the nonuniformness of γ.

Proof.[42] In (38),

$$\gamma^{1/2}\,dw = \gamma^{1/2} \circ dw - \frac{1}{2}\frac{\partial \gamma^{1/2}}{\partial x}\,dx\,dw$$

where the difference between $(\partial\gamma^{1/2}/\partial x) \circ dx$ and $(\partial\gamma^{1/2}/\partial x)\,dx$ is neglected because it is $O(dx\,dx\,dw)$ in the above expression. Then, setting $du = 0$ yields

$$
\begin{aligned}
u\,dt &= \frac{-1}{m\gamma}\frac{\partial\phi}{\partial x}dt + \frac{1}{\gamma}\circ\left(\gamma^{1/2}\circ dw - 1/2\frac{\partial\gamma^{1/2}}{\partial x}dx\,dw\right)\\
&= \frac{-1}{m\gamma}\frac{\partial\phi}{\partial x}dt + \gamma^{-1/2}\circ dw - \frac{1}{2\gamma}\frac{\partial\gamma^{1/2}}{\partial x}dx\,dw\\
&= \frac{-1}{m\gamma}\frac{\partial\phi}{\partial x}dt + \gamma^{-1/2}\,dw + \left(\frac{1}{2}\frac{\partial\gamma^{-1/2}}{\partial x} - \frac{1}{2\gamma}\frac{\partial\gamma^{1/2}}{\partial x}\right)dx\,dw\\
&= \frac{-1}{m\gamma}\frac{\partial\phi}{\partial x}dt + \gamma^{-1/2}\,dw - \frac{1}{2\gamma^{1/2}}\frac{\partial\log\gamma}{\partial x}dx\,dw
\end{aligned}
$$

which is set equal to dx in (39) in accordance with the prescription. This means that the process $x(t)$ is influenced by the white noise $w(t)$, so that $dx\,dw = O(dt)$, which is nonvanishing. This term can be calculated by inserting $\gamma^{-1/2}\,dw$ into dx, since $dx = \gamma^{-1/2}\,dw + O(dt)$. Thus, on replacing $(dw)^2$ by $(2kT/m)\,dt$ [which conforms to (41)],

$$
dx = \frac{-1}{m\gamma}\left(\frac{\partial\phi}{\partial x} + kT\frac{\partial\log\gamma}{\partial x}\right)dt + \gamma^{-1/2}\,dw \tag{43}
$$

The resulting Fokker–Planck equation by means of the Itô formula is identified with (42). QED.

A full mathematical justification of the above rule for the adiabatic reduction is not yet available, but at present we consider the prescription as a plausible way to obtain the steady-state distribution in terms of the reduced process. Accordingly, the multiplication of the white noise by the saturation factor $[1 + s(\nu)a^*a]^{-1}$ in the reduced process (35) is interpreted in the Stratonovich sense, as well as in

$$
\frac{dn}{dt} = 2\,\mathrm{Re}[\kappa + i(\omega - \nu)]\,a\alpha - Cn + \frac{An}{1 + s(\nu)n} + \mathscr{F}(t) \tag{44}
$$

where

$$
\mathscr{F}(t) = \left(a^*f + \frac{-ig}{\gamma_\perp}\frac{d(\nu)}{1 + s(\nu)n}a^*F + \text{c.c.}\right) + \frac{s(\nu)n}{1 + s(\nu)n}F_z \tag{45}
$$

Then, the "spurious drift" that arises when rewriting (45) in the Itô equation (such that $\langle\text{noise}\rangle = 0$) is shown to give a noise correction of (26) as follows:

$$
\begin{aligned}
\frac{dn}{dt} = {}& 2\,\mathrm{Re}[\kappa + i(\omega - \nu)]\,a\alpha - C(n - n_c) + \frac{A}{1 + s(\nu)n}(n - n_A)\\
&- \tfrac{1}{4}\gamma_{\|}s(\nu)\bar{M}\frac{s(\nu)n}{[1 + s(\nu)n]^2}
\end{aligned} \tag{46}
$$

with $n_A = \bar{M}/(-2Z_0) = O(1)$. Also, the correlation properties (36) yield

$$\langle \mathscr{F}^*(0)\mathscr{F}(t)\rangle = 2\left(Cn_c + \frac{|An_A|}{1 + s(\nu)n}\right)n\, d(t) \tag{47}$$

The Lasing State. In the absence of an external field, i.e., $\alpha = 0$, the phase diffusion can be separated, and

$$\partial p/\partial t = L_c p + L_A p \tag{48}$$

where

$$L_c p = \frac{\partial}{\partial n}\left[Cn_c n\left(\frac{1}{n_c}p + \frac{\partial p}{\partial n}\right)\right] \tag{49}$$

$$L_A p = \frac{\partial}{\partial n}\left[\frac{|An_A|n}{1 + s(\omega)n}\left(\frac{1}{n_A}p + \frac{\partial p}{\partial n}\right)\right] \tag{50}$$

Both (49) and (50) are of the form (42), allowing the steady-state distributions $L_c p = 0$ and $L_A p = 0$ given by the canonical ones e^{-n/n_c} and e^{-n/n_A}, respectively; for this reason the system can be adapted to the Lebowitz model of a two-reservoir open system. Above threshold for which $n_A < 0$, the steady-state solution $(L_c + L_A)p = 0$ for the lasing state is approximated by[4]

$$p_{st}(n) = \text{const} \times \exp\left[-\frac{s(\omega)C}{2A(n_c + |n_A|)}(n - n_s)^2\right] \tag{51}$$

We emphasize that this distribution, characteristic of the "far from equilibrium" situation, is a "synthesis" of the two canonical distributions e^{-n/n_c} and e^{-n/n_A} (see Fig. 3).

The Bistable Steady State of Absorption.[30] It is difficult to obtain the exact steady-state solution when $\alpha \neq 0$. However, for the resonance $\nu = \omega = \omega_0$, the reduced rate equation (23) has a fixed phase point $\phi = 0$, and the Fokker–Planck equation can be approximately given by

$$\partial p/\partial t = L_e p + L_d p \tag{52}$$

where

$$L_e p = -\frac{\partial}{\partial x}(\kappa y p) \qquad [x, y \text{ defined in (33)}] \tag{53}$$

$$L_d p = \frac{\partial}{\partial x}\left[D(x)\left(\frac{2x}{sn_c}p - \frac{1}{x}p + \frac{\partial p}{\partial x}\right)\right] \tag{54}$$

with

$$D(x) = \frac{\kappa s n_c}{2}\left(1 + \frac{2c}{1 + x^2}\right) \tag{55}$$

[4] This is an improved form over the one discussed in I.

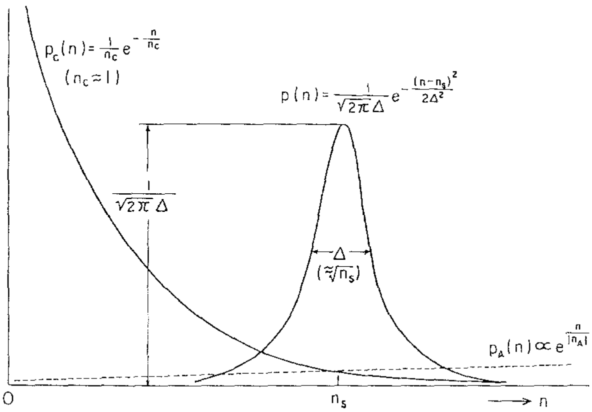

Fig. 3. A decomposition of the lasing distribution into two elementary distributions of canonical type. One component is actually *fictitious* because of its negative temperature; i.e., an unnormalizable distribution. This is of no harm for the existence of the lasing distribution, because only its logarithmic derivative is concerned.

Let us denote the steady-state solution $(L_e + L_d)p = 0$ by $p_y(x)$, which is expressed in terms of the potential $\Phi_y(x)$ as[5]

$$p_y(x) = \text{const} \cdot x \exp\left[-\frac{2}{n_c}\Phi_y(x)\right] = \text{const} \cdot x \exp\left[\int^x \frac{v_y(x')}{D(x')}\,dx'\right] \tag{56}$$

where

$$v_y(x) = j - \kappa\left(x + \frac{2cx}{1+x^2}\right), \qquad j = \kappa y \tag{57}$$

An elementary integration gives explicitly

$$\Phi_y(x) = \frac{1}{2s}(x-y)^2 + \frac{2cy}{s(2c+1)^{1/2}} \tan^{-1}\frac{x}{(2c+1)^{1/2}} \tag{58}$$

For vanishing external field, $y = 0$, the solution reduces to the canonical form $p_e(n) = \text{const} \cdot e^{-n/n_c}$, which represents blackbody radiation. The minima of the potential $\Phi_y(x)$ satisfy $(\partial/\partial x)\Phi_y(x) = 0$, which is identified with the nonlinear irreversibility relation (34). In the interval (y_m, y_M) it exhibits a clear double minimum and an unstable point (Fig. 4). Recently, there has been

[5] The same result has been obtained by Gragg *et al.*[43] from the population-dynamic master equation associated with (26).

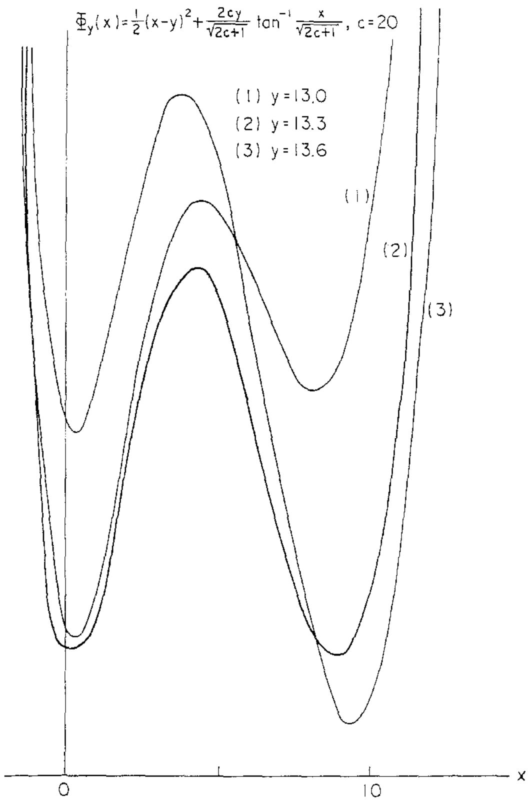

Fig. 4. Double minimum of the potential function $\Phi_y(x)$ defined in Eq. (58), which exhibits the bistable absorption.

considerable interest in the problem of multiple stability,[15,31–35] of which the explicit solution (58) offers a concrete example.

5. EFFECTS OF NOISE. II. EFFECTIVE TEMPERATURES

In their discussion of the laser linewidth in terms of the phase diffusion of the oscillating mode, Sargent *et al.*[9] introduced an effective negative temperature, to be interpreted as giving a negative value of the average photon number through the Planck formula $\bar{n}_m = [\exp(\hbar\omega_0/kT_m) - 1]^{-1}$.[9] This is precisely what we obtained in the preceding section as the noise correction of the photon number rate equation; viz. n_A given by (46). Therefore, we consider this concept as bearing further thermodynamic significance.

Let us rewrite the corrected equation (46), omitting the last term[6] of $O(sn)$:

$$\frac{dn}{dt} = 2\,\mathrm{Re}[\kappa + i(\omega - \nu)]\,a\alpha - C(n - n_c) + \frac{A}{1 + s(\nu)n}(n - n_A) \qquad (59)$$

This expression exhibits clearly the role of the noise corrections; each is a limit of the photon number n in contact with the respective reservoir, i.e., n_c with the cavity and n_A with the atoms. In such a consideration, the atomic degrees of freedom whose dynamical variables are represented by R and Z (the complex dipole and the population difference, see the Appendix) are incorporated into one of the reservoirs in contact with the oscillating mode of the field (the physical picture behind the adiabatic reduction of these variables), which provides another conceptual basis of the Lebowitz model. This then enables us to introduce a detailed balance condition: $n_c = n_A$ in the absence of external constraints, i.e., $\alpha = 0$ and $Z_0 =$ [equilibrium value], which yields the termwise satisfaction of the steadiness condition, $dn/dt = 0$.

The two Einstein relations for n_c and $\bar{M}$ in (36) tell us that

$$n_c = \langle b^*b\rangle_{\mathrm{eq}}\ (>0), \qquad n_A = \frac{1}{N_b - N_a}\langle R^*R\rangle_{\mathrm{eq}}\ (\gtrless 0) \qquad (60)$$

and the Planck formula for n_c is compatible with the detailed balance condition upon choosing $\langle R^*R\rangle_{\mathrm{eq}} = N_a$, which is satisfied by the sum of N independent Pauli spins. However, the expressions (60) are also compatible with detailed balance when the operator symmetrized form is assumed for b^*b and R^*R, for which

$$n_c = \tfrac{1}{2}\coth(\hbar\omega/2kT_c), \qquad n_A = \tfrac{1}{2}\coth(\hbar\omega_0/2kT_A) \qquad (61)$$

Here, the temperature T_A (identical with T_m in the literature[9,39]) is defined by $N_a = [\exp(-\hbar\omega_0/kT_A)]N_b$ in accord with the usual concept. Our discussion in subsequent sections will show how the expressions (61) play a thermodynamic role in the laser system.

[6] The omitted term $\frac{1}{2}\gamma_{\parallel}s(\nu)\bar{M}[1 + s(\nu)n]^{-2}[s(\nu)n]$ is $O(N^{-1})$ for the range $n = O(1)$ (i.e., of the order of the thermal fluctuations), which ensures that the noise correction deduced is the unique answer in the same range [provided the Stratonovich sense of the SD is the correct interpretation for the noise part in (45)]. When $n = O(N)$ (i.e., above threshold), the omitted term becomes of the same order as the n_c, n_A terms. But in such a situation, all the noise corrections become $O(N^{-1})$ of the main term and thus are insignificant. The same remark holds for writing the Fokker–Planck equations (48) and (52), where drift terms smaller than $O(N^{-1})$ of the main drift are discarded.

6. ENTROPY PRODUCTION AND THE MINIMUM PRINCIPLE

Lebowitz' formulation of the entropy production for a dissipative dynamical system[19,20,23] is based on the following two assumptions:

(a) If the system is in contact with a single, inexhaustible thermal reservoir characterized by a temperature $T = 1/k\beta$ and its approach to the equilibrium state is governed by a Markovian evolution law of the form $(d/dt)\rho = L\rho$, then the entropy production associated with the distribution ρ is given by

$$\sigma(\rho) = \frac{d}{dt} \operatorname{Tr} \rho(-\log \rho + \log \rho_e) = \operatorname{Tr}(L\rho)(-\log \rho + \log \rho_e) \tag{62}$$

$[\operatorname{Tr} \rho(d/dt) \log \rho = \operatorname{Tr} L\rho = 0]$, where the canonical equilibrium distribution $\rho_e = (1/Z_\beta)e^{-\beta H}$ satisfies

$$L\rho_e = 0 \tag{63}$$

(We have used the quantum terminology of *trace*, which will allow an appropriate classical interpretation.)

(b) Suppose that the system is in contact with more than two reservoirs, each of the above nature; the ith reservoir is characterized by $T_i = 1/k\beta_i$ and its contact with the system is described by the operator L_i,

$$L_i\rho_{e_i} = 0, \qquad \rho_{e_i} = (1/Z_{\beta_i})e^{-\beta_i H} \tag{64}$$

and further suppose that the time evolution of the system is governed by

$$d\rho/dt = L^{(H)}\rho + \sum_i L_i\rho \tag{65}$$

where $L^{(H)}$ is a purely mechanical Liouville operator with a Hamiltonian [which may be different from the H in (64)]. Then, the entropy production associated with any distribution ρ satisfying the evolution equation (65) is given by

$$\sigma(\rho) = \sum_i \sigma_i(\rho), \qquad \sigma_i(\rho) = \operatorname{Tr}(L_i\rho)(-\log \rho + \log \rho_{e_i}) \tag{66}$$

One can take into account the effect of thermal constraints by the unequal temperatures and the effect of mechanical constraints by the external field expressed in $L^{(H)}$.

A prototype of the entropy production associated with a system subject to the Fokker–Planck equation (42) is easily deduced: Let the equation be rewritten in the form

$$\frac{\partial p}{\partial t} = Lp = \frac{\partial}{\partial x}\left[D(x)p\left(\frac{\partial}{\partial x}\log\frac{p}{p_e}\right)\right], \qquad p_e(x) = \frac{1}{Z_\beta}e^{-\beta H(x)} \tag{67}$$

Then

$$\mathrm{Tr}(Lp)(-\log p + \log \rho_e) = -\int_{-\infty}^{\infty} \left\{\frac{\partial}{\partial x}\left[D(x)p\frac{\partial}{\partial x}\log\frac{p}{p_e}\right]\right\}\log\frac{p}{p_e}\,dx$$
$$= \int_{-\infty}^{\infty} D(x)\left(\frac{\partial}{\partial x}\log\frac{p}{p_e}\right)^2 p\,dx \geqslant 0 \qquad (68)$$

by the use of integration by parts and the boundary condition $p = 0$ at infinity. From (62), therefore, the expression (68) gives the rate of increase of the *relative entropy* $\int p(-\log p + \log p_e)\,dx$, showing that the approach of the system to equilibrium is subject to the second law of thermodynamics: The second representation in (62) allows us to interpret this rate as consisting of two parts: the rate of information entropy $S = \int p(-\log p)\,dx$ and a second part, due to the canonical form $\log p_e = -\beta H$, a heat flow divided by kT, so that

$$\sigma(p) = \int D(x)\left(\frac{\partial}{\partial x}\log\frac{p}{p_e}\right)^2 p\,dx = \frac{dS}{dt} - \frac{1}{kT}\int (Lp)H(x)\,dx \qquad (69)$$

The coefficient of $(kT)^{-1}$ in the last term must be interpreted as a heat gained by the system (flowing from the infinitely large heat reservoir), in view of the transformation from the Schrödinger to the Heisenberg picture:

$$\int (Lp)H(x)\,dx = \int p(x)(L^*H)(x)\,dx = \langle L^*H\rangle \qquad (70)$$

which implies the average of the rate of the system Hamiltonian over the distribution p.

Let us now obtain the entropy production $\sigma(p)$ for the many-reservoir open system under external fields governed by an evolution law of the type (65). Specifically, we consider the system governed by

$$L^{(H)}p = -\frac{\partial}{\partial x}[v(x)p], \qquad L_i p = \frac{\partial}{\partial x}\left[D^{(i)}(x)p\frac{\partial}{\partial x}\log\frac{p}{p_{e_i}}\right] \qquad (71)$$

where

$$p_{e_i}(x) = (1/Z_{\beta_i})e^{-\beta_i H(x)} \qquad (72)$$

and the "purely mechanical" nature of the operator $L^{(H)}$ is expressed as

$$\mathrm{div}\, v(x) = 0 \text{ — } v(x) = \text{const (one-dimensional case)} \qquad (73)$$

According to the Lebowitz formulation,

$$\sigma(p) = \sum_i \sigma_i(p) = \sum_i \int D^{(i)}(x)\left(\frac{\partial}{\partial x}\log\frac{p}{p_{e_i}}\right)^2 p\,dx$$
$$= \frac{dS}{dt} - \sum_i \frac{1}{kT_i}\langle L_i{}^*H\rangle \geqslant 0 \qquad (74)$$

where

$$S = \int p(-\log p)\,dx, \qquad \frac{dS}{dt} = \int (Lp)(-\log p)\,dx \tag{75}$$

L is given by the total evolution operator,

$$L = L^{(H)} + \sum_i L_i \tag{76}$$

Note that the condition (73) ensures

$$\int (L^{(H)}p) \log p \, dx = 0 \tag{77}$$

In the steady state p_{st} for which $Lp_{st} = 0$ formula (74) yields

$$\sigma(p_{st}) = -\sum_i \frac{J_i}{kT_i} \geqslant 0, \qquad J_i = \langle L_i{}^* H \rangle_{st} \tag{78}$$

The entropy production $\sigma(p_{st})$ does not vanish in general unless the external constraints disappear, i.e., $T_1 = T_2 = \cdots$ and $L^{(H)} = 0$. For example, when the formula is applied to the Kramers equation discussed in Section 2, the result agrees with (14) for N charged particles in a uniform electric field and with (21) for the same system but in contact with two reservoirs of temperatures T_1 and T_2.

In I, we have shown that there exists a variational principle which determines the steady-state solution p_{st}, $Lp_{st} = 0$, in the absence of an external field (i.e., $L^{(H)} = 0$). It involves minimizing a functional of two independent distributions p and $\hat{p}$, $\sigma(\hat{p}, p)$, with respect to one of them $\hat{p}$, requiring that the minimum be satisfied by the special condition $\hat{p} = p$, for which the minimal value of σ is identical with the entropy production $\sigma(p)$. This is a kind of self-consistency condition on the unknown distribution p that is shown to be equivalent to the steadiness condition $Lp = 0$. It can be extended to include the presence of an external field as follows.

Consider the functional defined by

$$\sigma(\hat{p}, p) = -2 \int v\left(\frac{\partial}{\partial x} \log \hat{p}\right) p\,dx + \sum_i \int D^{(i)}(x)\left(\frac{\partial}{\partial x} \log \frac{p}{p_{e_i}}\right)^2 p\,dx \tag{79}$$

It is lower bounded [i.e., $\sigma(\hat{p}, p) \geqslant \sigma_0 > -\infty$, *with* σ_0 a constant], and takes the minimal value, when varied with respect to $\hat{p}$, for each fixed p. Then the following two conditions are equivalent[24]:

(i) $$\hat{p}_{\min}(p) = p, \qquad \text{where} \quad \sigma[\hat{p}_{\min}(p), p] = \min_{\hat{p}} \sigma(\hat{p}, p) \tag{80}$$

(ii) $$Lp = -\frac{\partial}{\partial x}(vp) + \sum_i \frac{\partial}{\partial x}\left[D^{(i)}(x)\left(\frac{\partial}{\partial x} \log \frac{p}{p_{e_i}}\right)\right] = 0 \tag{81}$$

The proof is a straightforward extension of I, by noting that the additional term in the variation function (79) due to the external field depends on log $\hat{p}$ linearly, whereas the remainder has a quadratic dependence.

Here, we demonstrate how this variational principle can be used to obtain the steady-state distribution in practice. A simple example can be seen in free Brownian particles in contact with two reservoirs as discussed in Section 2:

$$\sigma(\hat{p}, p) = \int_{-\infty}^{\infty} \left[\frac{\gamma_1 k T_1}{m}\left(\frac{\partial}{\partial u}\log\frac{\hat{p}}{p_{e_1}}\right)^2 + \frac{\gamma_2 k T_2}{m}\left(\frac{\partial}{\partial u}\log\frac{\hat{p}}{p_{e_2}}\right)^2\right] p\, du \quad (82)$$

with

$$p_{e_i}(u) = \text{const}\cdot\exp[-\beta_i(mu^2/2)], \qquad \beta_i = 1/kT_i, \quad i = 1, 2$$

Let us choose the following form of the trial distributions p and $\hat{p}$:

$$\begin{aligned} p(u) &= (2\pi/m\beta)^{1/2}\exp[-\beta(mu^2/2)] \\ \hat{p}(u) &= (2\pi/m\hat{\beta})^{1/2}\exp[-\hat{\beta}(mu^2/2)] \end{aligned} \quad (83)$$

where β is an unknown parameter, and $\hat{\beta}$, also unknown, is the variation parameters. A direct integration of the right-hand side of (82) yields an expression to be minimized with respect to $\hat{\beta}$:

$$\sigma(\hat{p}, p) = \frac{N}{\beta}\left[\frac{\gamma_1}{\beta_1}(\hat{\beta} - \beta_1)^2 + \frac{\gamma_2}{\beta_2}(\hat{\beta} - \beta_2)^2\right] \quad (84)$$

and set $\hat{\beta} = \beta$ for self-consistency. This provides the expected result:

$$T\left(=\frac{1}{k\beta}\right) = \frac{\gamma_1 T_1 + \gamma_2 T_2}{\gamma_1 + \gamma_2}, \qquad \sigma(p_{\min}) = \frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2}\frac{N}{T_1 T_2}(T_2 - T_1)^2$$

in agreement with (19) and (21), respectively.

It can be observed in the above demonstration why the condition of self-consistency is necessary to get the exact distribution by the variational principle: one might be tempted to set up the variational principle directly by minimizing the functional $\sigma(p)$ with respect to the normalized distribution p to get the result, as motivated by the thermodynamic principle of "minimal entropy production." If this is applied to the above simple example, (84) is then replaced by

$$\sigma(p) = \frac{N}{\beta}\left[\frac{\gamma_1}{\beta_1}(\beta - \beta_1)^2 + \frac{\gamma_2}{\beta_2}(\beta - \beta_2)^2\right]$$

to be minimized with respect to β. This does not give the right result, due to the extra factor of variation β^{-1} in front of the bracket. Clearly, this factor arises from the distribution p over which the remaining quadratic quantities are averaged to evaluate $\sigma(p)$.

Therefore, the self-consistency condition tells us the two different roles of the distribution p in the functional by denoting them as $\hat{p}$ and p; viz. the *varying* distribution $\hat{p}$ and the *averaging* distribution p, the latter not being subject to the variation: $\hat{p}$ is considered as the distribution containing the effect of fluctuations acting on p, and the variation is to be made against such fluctuations (the spirit of "local potential"[36,37]).

This variational principle can be used for more complex nonlinear problems in accordance with (80) and (81). In a separate paper we show that the laser distribution (51) is indeed a consequence of it (by a trial distribution of the form $p(n) \propto \exp[-\beta(n-n_s)^2]$ with two variation parameters β and n_s): the distribution is synthesized as the *optimum* of two elementary distributions of thermal equilibrium type, and the present variational principle shows just how to optimize them.

7. THERMODYNAMIC RELATIONS

Suppose that a dissipative dynamical system is subject to an evolution equation of the standard form (65), expressed in terms of Fokker–Planck operators on its probability distribution p, i.e.,

$$\frac{\partial p}{\partial t} = L^{(H)}p + \sum_i L_i p, \qquad L^{(H)}p = -\frac{\partial}{\partial x_\mu}(v_\mu p)$$

($\operatorname{div} v = \partial v_\mu/\partial x_\mu = 0$), and

$$L_i p = \frac{\partial}{\partial x_\mu}\left[D^{(i)}_{\mu\nu}(x)p\,\frac{\partial}{\partial x_\nu}\left(\log p + \frac{H(x)}{kT_i}\right)\right] \tag{85}$$

(the usual summation convention for repeated indices is implied). Then, the first and second laws of thermodynamics,

$$dE = \delta W + \sum_i \delta Q_i \qquad \text{(plus sign for a gain of the system)}$$

$$dS \geqslant \sum_i \frac{1}{kT_i}\,\delta Q_i \qquad \text{(entropy in dimensionless units)}$$

can be identified in the statistical mechanical model (85) with the following expressions: use the notation for an average $\langle X\rangle = \int X(x)p\,dx$, and $E = \langle H\rangle$ and $S = \langle(-\log p)\rangle$ for the internal energy and the entropy, respectively. In a time interval dt (on a macroscopic scale)

$$dE = dt\,\langle L^* H(x)\rangle \tag{86}$$

$$dS = dt\,\langle(-L^* \log p)\rangle \tag{87}$$

where the total differential d is assigned only to the total evolution operator $L^* = L^{(H)*} + \sum_i L_i^*$ given by

$$L^{(H)*}X = v_\mu \frac{\partial X}{\partial x_\mu} \tag{88}$$

$$L_i^*X = \frac{\partial}{\partial x_\mu}\left(D_{\mu\nu}^{(i)} \frac{\partial X}{\partial x_\nu}\right) - \frac{D_{\mu\nu}^{(i)}}{kT_i}\frac{\partial H}{\partial x_\mu}\frac{\partial X}{\partial x_\nu} \tag{89}$$

For the other component differentials the notation δ is used and

$$\delta W = dt\,\langle L^{(H)*}H\rangle = dt\left\langle v_\mu \frac{\partial H}{\partial x_\mu}\right\rangle \tag{90}$$

$$\delta Q_i = dt\,\langle L_i^*H\rangle = dt\left\langle \frac{\partial}{\partial x_\mu}\left(D_{\mu\nu}^{(i)} \frac{\partial H}{\partial x_\nu}\right) - \frac{D_{\mu\nu}^{(i)}}{kT_i}\frac{\partial H}{\partial x_\mu}\frac{\partial H}{\partial x_\nu}\right\rangle \tag{91}$$

We now discuss the application of this to our optical system.

7.1. The Lasing State

The state of affairs where a self-oscillation is taking place but the power it generates is still confined inside the cavity (of Fabry–Perot type) can be represented by the evolution equation (48) with the Liouvillian $L = L_c + L_A$ given by (49) and (50), i.e., $\delta W = 0$. At the steady state $dE = dS = 0$, so that the two thermodynamic relations become

$$0 \geqslant \left(\frac{1}{k\bar{T}_A} - \frac{1}{k\bar{T}_c}\right)\delta Q_A, \qquad \delta Q_c = -\delta Q_A \tag{92}$$

where

$$k\bar{T}_c \equiv \frac{\hbar\omega}{2}\coth\frac{\hbar\omega}{2kT_c}, \qquad k\bar{T}_A \equiv \frac{\hbar\omega_0}{2}\coth\frac{\hbar\omega_0}{2kT_A} \tag{93}$$

in accordance with (61). Thus, positive T_c and negative T_A above threshold indicates merely $\delta Q_A > 0$ (heat flows from the atoms). However, this is just a realization of the ideal heat engine: it takes heat from a reservoir of higher temperature and supplies heat to another reservoir of lower temperature, and the negativeness of the higher temperature admits the possibility that the efficiency of the engine can be 100% without violating the second law, provided the original formulation is properly revised (a new type of "perpetual motion"[4,6]). This can be seen by introducing the work $-\delta W$ in the energy conservation relation, i.e., $\delta Q_c = -\delta Q_A + \delta W$, in place of (92) and substituting it into the inequality of the second law, obtaining that

$$\eta \equiv \delta W/\delta Q_A \leqslant 1 - \bar{T}_c/\bar{T}_A < 1 \qquad \text{for} \quad \bar{T}_A > 0 \tag{94}$$

but the negativeness of $\bar{T}_A$ makes it possible that the maximum value of η can exceed unity. It actually takes a value >1 provided $\delta Q_c > 0$ (corresponding to the situation that the engine takes heat from both reservoirs, possibly cooling the lower temperature reservoir). Even if δQ_c is restricted to negative values, η can be made arbitrarily near to unity by reducing the dissipative loss δQ_c.

A simple expression of the efficiency of the laser in the present model can be obtained under the assumption that the energy flow from inside to outside of the cavity is expressed by an additional drift velocity $-Bn$ in the Fokker–Planck equation (48), where B is a positive constant determined by the structure of the cavity (it plays the role of a sink with respect to the power generated inside the cavity without dissipation). This yields $\eta = B/(B + C)$, which agrees with what is known from laser technology.[38,39] A precise determination of the efficiency requires the conversion ratio of the work versus the power so emitted, which can be unity only for ideally coherent laser light.[7]

7.2. Bistable Steady State of Absorption[7]

The system can be modeled by an assembly of gas atoms inside a ring cavity irradiated by coherent laser light, which is reemitted,[31] sometimes called resonant fluorescence, accompanied by absorption; this is a typical nonlinear response phenomenon. Let it be described by the simplified Fokker–Planck equation (52) with (53)–(55). The dissipative part L_d in (54) corresponds to $L_c + L_A$ in the laser, but here represents a system in contact with a single reservoir due to the condition of detailed balance $T_A = T_c$. The thermodynamic relation for the steady state is accordingly given by

$$0 \geqslant -\delta W/\bar{T}_c, \qquad \delta W = -(\delta Q_c + \delta Q_A) \tag{95}$$

indicating that $\delta W > 0$.

The quantity $\bar{T}_c^{-1}\,\delta W$ in the above relation represents the entropy production at the steady state (for a time interval dt), which can be calculated either by formula (90) or by a formula of the form of (69), i.e.,

$$\sigma(p_{st}) = \int_0^\infty D(x)\left(\frac{\partial}{\partial x}\log\frac{p_{st}}{p_e}\right)^2 p_{st}\,dx \tag{96}$$

with $p_{st}(x) = p_y(x)$ given by (56)–(58) and $p_e(x) = p_{y=0}(x)$. The result is as follows:

$$\sigma(p_{st}) = \langle j^2/D(x)\rangle = (2/sn_c)j\langle x\rangle \tag{97}$$

[7] Recently, a new problem of "chaotic" steady states has arisen with regard to dispersive effects.[41]

where x and $y = \kappa^{-1}j$ are the amplitude of the total field and the external field, respectively, scaled as in (33), and $D(x)$ is the nonconstant diffusion coefficient of the Fokker–Planck equation (52), given by

$$D(x) = \frac{\kappa s n_c}{2}\left(1 + \frac{2c}{1 + x^2}\right) \tag{98}$$

Clearly, the first term in the parentheses 1 corresponds to absorption of the total field by the cavity wall and the second $2c(1 + x^2)^{-1}$ to absorption by the atoms, whose relative importance determines the bistable points; its nonconstancy is therefore linked intimately with the potential Φ_y, (58), in the steady-state distribution: The problem of relative stability, i.e., the determination of the allowed value x of the multiple roots of the nonlinear relation (34) for a given y, can be solved by finding the absolute minimum of the potential $\Phi_y(x)$, as noted by several authors,[31–33] and therefore the phase transition at which the discontinuity of x occurs is identified with y_0; the coincidence point of the double minimum of the potential Φ_y is such that

$$\int_{x_1}^{x_3}\left(x + \frac{2cx}{1 + x^2} - y_0\right)\frac{dx}{D(x)} = 0, \qquad c > 4 \tag{99}$$

which is equivalent to the Maxwell construction of the transition point (see Fig. 5). Hence, the averaged quantity $\langle x \rangle$ is a single-valued function of y analytic except in the vicinity of y_0, where a discontinuity takes place, the discontinuity being rigorous in the thermodynamic limit, $N_s = s^{-1} \to \infty$.

This discontinuity is precisely what arises in the second representation of the entropy production given by (97), proportional to $j\langle x \rangle$: It justifies the conventional writing of the entropy production as [flux] times [force] in the sense of an ensemble average over the steady-state distribution far from equilibrium, where the average of the [force] is a nonlinear (discontinuous) function of the [flux]. Indeed, this nonlinear flux–force relation is given by the following identity due to the two representations in (97):

$$\langle x \rangle = \frac{1}{\kappa}\left\langle \left(1 + \frac{2c}{1 + x^2}\right)^{-1} \right\rangle j \tag{100}$$

[cf. (34′)], yielding the rigorous meaning of the Onsager coefficient in this nonlinear irreversibility. We remark that the entropy production so established has no meaning with regard to the indication of the relative stability: At the transition point y_0, the lower value of $\langle x \rangle$ is more stable for $y < y_0$, but the higher value of $\langle x \rangle$ is more stable for $y > y_0$. Instead, the averaged quantity $\langle x \rangle$ in the entropy production plays the role of order parameter in such a dissipative phase transition, indicating the existence of an ordered phase in the lower branch where the entropy production is anomalously small.

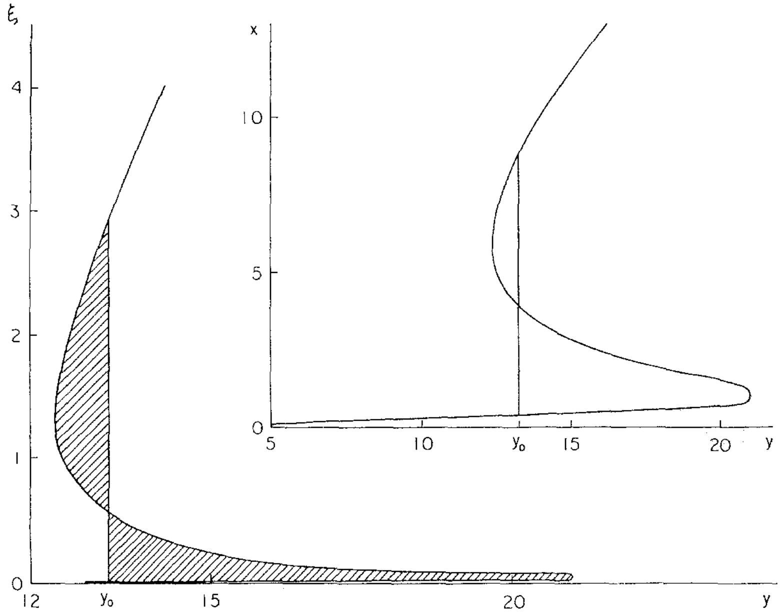

Fig. 5. Ensemble average $\langle x \rangle$ over the steady-state distribution (56) with (58), vs. y. The point of discontinuity y_0 can be determined by the "equal area" rule in the scaled coordinate $\xi = \int_0^x \{1 + 2c/[1 + (x')^2]\}^{-1} \, dx'$ (the two hatched regions are equal in area).

8. DISCUSSION

A basic issue that cannot be overlooked for the present approach is Van Kampen's criticism of the use of the nonlinear Langevin equations. Actually, his criticism extends to the possibility of describing physical processes of a stochastic nature by means of the Kolmogorov equations (which by itself is mathematically acceptable, see Ref. 35, discussion part), and so we divide the criticism into two claims. The first is the claim[17] that real errors can arise if a Langevin equation with a nonlinear deterministic part is constructed just by supplementing it by a Gaussian white noise term with a constant strength. Our treatment of the laser Langevin equation (35) is free from this danger, in view of its derivation from the starting simultaneous Langevin equations on the basis of the stochastic calculus. We point out that it is necessary to establish a mathematical foundation of the method of adiabatic reduction including noise terms: Otherwise, the prescription discussed in Section 4 cannot be put on an entirely solid basis. We also note that the starting set of the Langevin equations itself should be subject to this warning, but we have a sound basis in the Brownian motions of photons and spins discussed in I.

The second claim is the assertion that, from the standpoint of the so-called Ω-expansion of the master equation, general diffusion processes with nonlinear drift velocities and nonconstant diffusion coefficients as functions of the state variables are possible only under the special circumstance that the first term of the Ω-expansion vanishes identically: otherwise, a consistent description in terms of second-order differential operators should be restricted only to those with linear drift velocities and constant diffusion coefficients (though they may be complex functions of time), and if one wishes to go beyond this stage of approximation it is necessary to take into account higher order derivatives for the consistency of the expansion scheme.[18] This is a rather strict statement, which our laser Langevin equation and hence the corresponding diffusion (Fokker–Planck) equation really do not fit. [Note that in our laser system the expansion parameter Ω is given by N, i.e., the total number of atoms, and by this interpretation the laser Langevin equation (35) is classified into the *third* case[18] of Van Kampen, where he rejects going beyond the linear fluctuation in the vicinity of the stationary point within the "diffusion approximation".]

We have no further comment on Van Kampen's criticism, but maintain the plausibility of our standpoint. We note in this respect that there exists another scheme of expanding and truncating the quantum master equation for a laser which reduces to the laser Fokker–Planck equation (48) and which also takes into account the essential quantum fluctuation effect represented in formula (61) (see the Appendix of I).

APPENDIX. LASER RATE EQUATION FOR TWO-LEVEL ATOMS

Let us denote the five random variables of our optical system, as in I, as follows: b, b^* are the complex amplitudes of the mode of the em field, which may grow; R, R^* are the complex amplitudes of the atomic dipole collective mode; Z is the total atomic population difference $= \frac{1}{2}(N_a - N_b)$, where a denotes the upper and b the lower level, respectively.

Further, we assume that the system is irradiated by a coherent external field α_t, which is added to the mode amplitude b. Then, it is plausible to describe the purely mechanical motion of the system by means of a Hamiltonian given in the rotating field approximation by

$$H = \hbar\omega b^*b + \hbar\omega_0 Z + \hbar g[(b^* + \alpha^*)R + (b + \alpha)R^*] \tag{A1}$$

where the frequency ω is associated with the mode, ω_0 with the dipole, and g with the interaction between the dipole and the total field. (In I, we used the notation $\lambda/\sqrt{N}$ for g to indicate that this coupling constant is $O(N^{-1/2})$, where N is the total number of atoms in the cavity.) The equations of motion for these variables can be obtained by using the commutation relations $[b, b^*] = 1$, $[Z, R^*] = R^*$, and $[Z, R] = -R$ in the quantum mechanical analog of a harmonic oscillator and an angular momentum vector, respectively, of the system under consideration in a classical mechanical framework:

$$\begin{aligned} \dot{b} = -i\omega b - igR, \qquad \dot{R} = -i\omega_0 R + 2igZ(b + \alpha_t) \\ \dot{Z} = ig[(b^* + \alpha_t^*)R - (b + \alpha_t)R^*] \end{aligned} \tag{A2}$$

The effect of the thermal environment on the system can be taken into account by supplementing each equation by a systematic damping term and a residual noise term, i.e., by making it a Langevin equation. The most convenient way to do this without losing internal consistency is to adopt the model of linear Brownian motion for each dynamical species, taking (b, b^*) and (R, R^*, Z) in the absence of coupling, yielding the picture of Brownian motion of photons and that of spins, respectively, as discussed in I. This is still valid in the presence of the external field α_t, where it is assumed simply not to be subject to the damping and noise. Then, it can be shown that a consistent choice of the field variable is not b but the total field a defined by

$$a \equiv b + \alpha_t \tag{A3}$$

The set of Langevin equations is written, accordingly, as follows:

$$\begin{aligned} \dot{a} &= -(\kappa + i\omega)(a - \alpha_t) + \dot{\alpha}_t - igR + f, \qquad \dot{a}^* = \text{c.c.} \\ \dot{R} &= -(\gamma_\perp + i\omega_0)R + i2gZa + F, \qquad \dot{R}^* = \text{c.c.} \\ \dot{Z} &= -\gamma_\parallel(Z - Z_0) + ig(a^*R - aR^*) + F_z \end{aligned} \tag{A4}$$

where the usual white noise character and the Einstein relation will be assumed for the noise part. [The reason for the choice of a for the field variable rather than b is that by this choice the role of external field is expressed as a sourceless drift velocity—its divergence vanishing—in the Fokker–Planck framework associated with (A4) as well as with its adiabatic reduction, as discussed in Section 6.]

It is convenient to reexpress Eq. (A4) in the rotating frame with (arbitrary) frequency ν, i.e., in terms of the transformed variables $\bar{a} = ae^{i\nu t}$ and $\bar{R} = Re^{i\nu t}$:

$$\begin{aligned} \dot{\bar{a}} &= -[\kappa + i(\omega - \nu)](\bar{a} - \bar{\alpha}_t) + (i\nu + \dot{\alpha}_t/\alpha_t)\bar{\alpha}_t - ig\bar{R} + \bar{f} \\ \dot{\bar{R}} &= -[\gamma_\perp + i(\omega_0 - \nu)]\bar{R} + i2gZ\bar{a} + \bar{F}, \qquad \text{c.c.} \\ \dot{Z} &= -\gamma_\parallel(Z - Z_0) + ig(\bar{a}^*\bar{R} - \bar{a}\bar{R}^*) + F_z \end{aligned} \tag{A4$'$}$$

where the frequency ν is determined best for the steady situation. Suppose that the external field is harmonic with a single frequency ω_1 $[\alpha_t = \alpha \exp(-i\omega_1 t)]$ and the field is induced by this external field. Then, in the steady state, $\nu = \omega_1$ should be the appropriate choice for which the term $(i\nu + \dot{\alpha}_t/\alpha_t)\bar{\alpha}_t$ vanishes and the resulting deterministic part of (A4′) becomes autonomous. However, if the system becomes self-oscillating (lasing), a proper frequency of the oscillation is determined, which is given in the present model by

$$\nu = (\kappa\omega_0 + \gamma_\perp\omega)/(\kappa + \gamma_\perp) \tag{A5}$$

This frequency makes the imaginary part of the secular determinant for $\bar{a}$ and $\bar{R}$ on the right-hand side of $\dot{\bar{a}}$ and $\dot{\bar{R}}$ in (A4′) vanish. Under this lasing circumstance, the real part of the determinant also vanishes, the threshold value of Z being attained, so that a linear relation is formed between $\bar{a}$ and $\bar{R}$. This is given either by

$$\bar{R} = \frac{(1 + \kappa/\gamma_\perp)2igZ_{\text{th}}}{\kappa + \gamma_\perp - i(\omega - \omega_o)}\,\bar{a} \tag{A6}$$

or

$$\bar{a} = \frac{-(1 + \gamma_\perp/\kappa)ig}{\kappa + \gamma_\perp - i(\omega - \omega_0)}\,\bar{R} \tag{A6′}$$

where

$$Z_{\text{th}} = \frac{\kappa\gamma_\perp}{2g^2}\left[1 + \left(\frac{\omega - \omega_0}{\kappa + \gamma_\perp}\right)^2\right] \tag{A7}$$

We now present an argument to justify the "adiabatic elimination."

Suppose the situation $\kappa \ll \gamma_\perp$. Then, $\nu \simeq \omega$ by virtue of (A5) and, at the same time, $\omega_0 - \nu \simeq -(\omega - \omega_0)$, so that the relation (A6) just arises from $\dot{\bar{R}} = 0$ in (A4′), where the noise part is disregarded. [Similarly, in the opposite case, $\kappa \gg \gamma_\perp$, $\nu \simeq \omega_0$ and $\omega - \nu \simeq \omega - \omega_0$, for which the relation (A6′) is the consequence of $\dot{\bar{a}} = 0$. We do not go into this situation.] This suggests that we choose the frequency ν of the rotating frame equal to ω, the frequency of the mode, and eliminate $\bar{R}$ by substituting expression (A6) $(\kappa \ll \gamma_\perp)$ into the other two equations. For the $\dot{Z}$ part, this gives

$$\dot{Z} = -\gamma_\parallel(Z - Z_0) - \gamma_\parallel s(\omega)Zn \tag{A8}$$

and the threshold condition that is obtained in combination with the other equation $\dot{n}$ is consistent with (A7). Therefore near the steady state, it further suggests that we eliminate Z to get a single equation for $\dot{a}$ from

$$\dot{Z} = 0, \qquad Z = Z_0/[1 + s(\omega)n] \tag{A9}$$

where

$$s(\omega) = (4g^2/\gamma_\parallel\gamma_\perp)/\{\gamma_\perp{}^2/[\gamma_\perp{}^2 + (\omega - \omega_0)^2]\} \tag{A10}$$

Actually, the procedure of adiabatic elimination is expected to have a wider applicability than the above discussion, provided that the inequalities on the basis of which Eq. (23) is derived, $\kappa, |\nu - \omega| \ll \gamma_{\perp}, \gamma_{\parallel}$, are satisfied. Also, Eq. (35) is the formal application of the same procedure to include the noise part in (A4′).

NOTE ADDED IN PROOF

After submitting this paper, we have carried out a detailed analysis of the adiabatic reduction of fast variables from a set of Langevin equations, obtaining a result that indicates the full validity of the reduced Fokker–Planck operators given by (50) and (54), but also the necessity of a certain modification of the prescriptions (I)–(IV) in Section 4. This means that the form (46), derived when (I)–(IV) are literally applied to the starting laser Langevin equation (A4), is incorrect. Accordingly, footnote 6 becomes a superfluous statement. This matter will be reported separately [see also Ref. (42) and (44)].

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REFERENCES

1. H. Hasegawa and T. Nakagomi, *J. Stat. Phys.* **21**:191 (1979).
2. H. Haken, *Synergetics—An Introduction* (Springer-Verlag, Berlin, 1978).
3. H. Haken, *Coherence and Quantum Optics IV*, L. Mandel and E. Wolf, eds. (Plenum Press, New York, 1978), p. 49.
4. M. Marvan, *Negative Absolute Temperatures* (Iliffe, London, 1966).
5. V. K. Konyukhov and A. M. Prokhrov, *Sov. Phys.—Usp.* **19**:613 (1976).
6. P. T. Landsberg, *J. Phys. A* **10**:1773 (1977).
7. M. Garbuny, *J. Chem. Phys.* **67**:5676 (1977).
8. T. Nakagomi, *J. Phys. A* **13**:291 (1980).
9. M. Sargent, M. O. Scully, and W. E. Lamb, *Laser Physics* (Addison-Wesley, London, 1974).
10. R. Bonifacio and L. A. Lugiato, *Opt. Commun.* **19**:172 (1976); *Phys. Rev. Lett.* **40**:1023 (1978).
11. H. J. Carmichael and D. F. Walls, *J. Phys.* **10B**:685 (1977); *Prog. Theor. Phys. Suppl.* **64**:307 (1978).
12. R. Landauer, *J. Appl. Phys.* **33**:2209 (1962).
13. R. Landauer and J. W. F. Woo, *Statistical Mechanics: New Concepts, New Problems, New Applications*, S. A. Rice, K. T. Freed, and J. C. Light, eds. (University of Chicago Press, 1972), p. 299.

14. R. Landauer, *J. Stat. Phys.* **13**:1 (1975).
15. J. W. F. Woo and R. Landauer, *IEEE J. Quantum Electronics* 7:435 (1971).
16. R. Landauer, *Phys. Rev. A* **18**:225 (1978).
17. N. G. Van Kampen, *Adv. Chem. Phys.* **34**:245 (1976).
18. N. G. Van Kampen, *Phys. Lett.* **62A**:383 (1977).
19. J. L. Lebowitz, *Phys. Rev.* **114**:1192 (1959).
20. P. G. Bergmann and J. L. Lebowitz, *Phys. Rev.* **99**:578 (1955).
21. J. L. Lebowitz and P. G. Bergmann, *Ann. Phys. (N.Y.)* **1**:1 (1957).
22. J. L. Lebowitz and A. Shimony, *Phys. Rev.* **128**:1945 (1962).
23. H. Spohn and J. L. Lebowitz, *Adv. Chem. Phys.* **38**:109 (1978).
24. H. Hasegawa and T. Nakagomi, "On the characterization of the stationary state of a class of dynamical semigroups," *J. Stat. Phys.* (to be published).
25. G. E. Uhlenbeck and L. S. Ornstein, *Phys. Rev.* **36**:823 (1930).
26. S. Chandrasekar, *Rev. Mod. Phys.* **15**:1 (1943).
27. M. C. Wang and G. E. Uhlenbeck, *Rev. Mod. Phys.* **17**:323 (1945).
28. S. R. De Groot and P. Mazur, *Nonequilibrium Thermodynamics* (North-Holland, Amsterdam, 1962).
29. D. Ryter, *Z. Physik B* **30**:219 (1972).
30. K. Kondo, M. Mabuchi, and H. Hasegawa, *Opt. Commun.*, **32**:136 (1980).
31. R. Bonifacio, M. Gronchi, and L. A. Lugiato, *Phys. Rev. A* **18**:2266 (1978).
32. I. Matheson, D. F. Walls, and C. W. Gardiner, *J. Stat. Phys.* **12**:21 (1975).
33. I. Procaccia and J. Ross, *J. Chem. Phys.* **67**:5565 (1977); *Prog. Theor. Phys. Suppl.* **64**:244 (1978).
34. A. Schenzle and H. Brand, *Opt. Commun.* **27**:485 (1978); *Phys. Rev.*, to appear.
35. N. G. Van Kampen, *Prog. Theor. Phys. Suppl.* **64**:389 (1978).
36. P. Glansdorff and I. Prigogine, *Thermodynamic Theory of Structure, Stability and Fluctuations* (Wiley—Interscience, London, 1971).
37. H. Hasegawa, *Prog. Theor. Phys.* **58**:128 (1977); *Prog. Theor. Phys. Suppl.* **64**:321 (1978).
38. K. Shimoda and T. Yajima, eds., *Quantum Electronics* (1972) (in Japanese).
39. A. Yariv, *Quantum Electronics* (Wiley, New York, 1967).
40. R. L. Stratonovich, *Conditional Markov Processes and Their Application to Optimal Control* (Elsevier, New York, 1968).
41. K. Ikeda, *Opt. Commun.* **30**:257 (1979).
42. H. Hasegawa, M. Mabuchi, and T. Baba, *Phys. Lett.* (to appear).
43. R. F. Gragg, W. C. Schieve, and A. R. Bulsra, *Phys. Rev. A* **19**:2052 (1979).
44. H. Hasegawa, in *Relaxation of Elementary Excitations, Solid State Sciences 18*, R. Kubo and E. Hanamura, eds. (Springer-Verlag, New York, 1980).